TIME-ORDERED PRODUCTS AND *MATHEMATICA*

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We will be discussing a method of computing the integral curves of time-varying vector fields. Physically, we can picture these as the paths taken by small parcels of water in a changing current. We will start with a few examples in which the vector field takes a simple form, and then arrive at a general solution that can be applied to a class of such fields.

If $\mathbf{V}$ is a vector field in space (*i.e.* $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$), then an *integral curve* is a parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ in space having the property that its tangent vector always matches $\mathbf{V}$. To arrive at an equation, we note that at time t, the tangent to the curve γ is $\frac{d\gamma}{dt}(t)$, and the vector field evaluated at the point $\gamma(t)$ is $\mathbf{V}(\gamma(t))$. Along the path of the integral curve, these are required to agree for every time t:

$$\frac{d\gamma}{dt}(t) = \mathbf{V}(\gamma(t)). \tag{1}$$

A good reference on the topic of integral curves, generally, is *cf.* [ZT], Chapters II and III. A nice first example comes from the Larmor equation, which describes the precession of the magnetic moment of an atomic nucleus under the influence of a constant magnetic field $\mathbf{B}$, [GSU]:

$$\frac{d\gamma}{dt} = \mathbf{B} \times \gamma, \quad \gamma(0) = \gamma_0. \tag{2}$$

Writing $\mathbf{B} = \langle a, b, c\rangle$ and $\gamma(t) = \langle x(t), y(t), z(t)\rangle$ we compute

$$\mathbf{B} \times \gamma = \langle bz - cy, cx - az, ay - bx\rangle$$

obtaining the system of ODEs

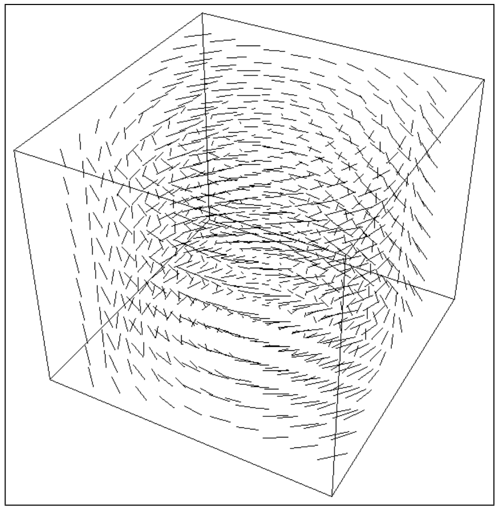
$$\frac{d\gamma}{dt} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tilde{\mathbf{B}}\gamma$$

where the last equality defines $\tilde{\mathbf{B}}$.

We compute the integral curves γ of the vector field $\mathbf{V}(\gamma) = \mathbf{B} \times \gamma = \tilde{\mathbf{B}}\gamma$.
In the special case that $\mathbf{B} = \langle 0, 0, 1\rangle$, equation (2) takes the form

$$\gamma' = \langle -y, x, 0\rangle.$$

The vector field looks like this:



so we see that the current moves in horizontal circles. Actually, since we may always change coordinates so that $\mathbf{B}$ points in the positive z direction, this special case describes fully the solution of the Larmor equation for constant $\mathbf{B}$.

Whenever (1) can be written

$$\frac{d\gamma}{dt}(t) = \mathbf{V}(\gamma(t)) = \mathbf{A}\gamma \tag{3}$$

where $\mathbf{A}$ is a general, constant matrix, we argue as in Differential Equations. There, we were able to give meaning to expressions like e^{it}:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \rightsquigarrow e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \rightsquigarrow e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \cos(t) + i\sin(t),$$

in order to solve equations like $y'' + y = 0$. We guess analogously that

$$\gamma(t) = e^{t\mathbf{A}}\gamma_0 \text{ is the solution to } \gamma' = \mathbf{A}\gamma,\ \gamma(0) = \gamma_0. \tag{4}$$

Using the power series representation of the exponential, we may exponentiate any*thing* that can be taken to powers, multiplied by rational constants, and summed, as long as we are able to take limits of these *things* too. In other words, if every ingredient in $\sum_{k=0}^{\infty} x^k/k!$ makes sense for x in some set (a topological algebra), then the exponential makes sense. Here, we will have $x = \mathbf{A}$ be *any* square matrix of complex numbers. We can prove that $\gamma(t) = e^{t\mathbf{A}}\gamma_0$ is the solution to the initial value problem $\gamma' = \mathbf{A}\gamma, \gamma(0) = \gamma_0$ by writing out the power series and differentiating formally. Convergence can be verified with the ratio test. In the case of the Larmor precession, for example, if $\mathbf{B} = \langle a, b, c\rangle$, and $\mathbf{B} \times \gamma = \tilde{\mathbf{B}}\gamma$, then setting $\rho^2 = a^2 + b^2 + c^2$

$$e^{\tilde{\mathbf{B}}} = \frac{1}{\rho^2}\begin{pmatrix} a^2 + (b^2+c^2)\cos(\rho) & ab(1-\cos(\rho)) - c\rho\sin(\rho) & ac(1-\cos(\rho)) + b\rho\sin(\rho) \\ ab(1-\cos(\rho)) + c\rho\sin(\rho) & b^2 + (a^2+c^2)\cos(\rho) & bc(1-\cos(\rho)) - a\rho\sin(\rho) \\ ac(1-\cos(\rho)) - b\rho\sin(\rho) & bc(1-\cos(\rho)) + a\rho\sin(\rho) & c^2 + (a^2+b^2)\cos(\rho) \end{pmatrix}.$$

Obviously, this computation would be extremely tedious to perform. I did it using the *Mathematica* command

`MatrixExp[]`.

Taking $\mathbf{B} = \langle 0, 0, 1\rangle$, we get

$$e^{t\tilde{\mathbf{B}}} = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $e^{t\tilde{\mathbf{B}}}\gamma_0$ gives us the horizontal circles we expect.

Now what happens if $\mathbf{A}$ is an arbitrary matrix varying in time? Emboldened as we are, we may be tempted to proceed as in our Differential Equations class and use an integrating factor:

$$y' - p(t)y = 0,\ y(0) = y_0 \rightsquigarrow [e^{-\int^t p}y]' = 0 \Rightarrow e^{-\int^t p}y = C \Rightarrow y(t) = Ce^{\int^t p} \Rightarrow y(t) = e^{\int^t p}y_0,$$

so we try to build $e^{\int^t \mathbf{A}(t)dt}$. This is rather pretty as it reduces to the right formula when $\mathbf{A}$ happens to be a constant. This approach is doomed to fail when $\mathbf{A}$ is non-constant. The reason is that it could happen that $\mathbf{A}(t)\mathbf{A}(t') \neq \mathbf{A}(t')\mathbf{A}(t)$ when $t \neq t'$. For example, even for vector fields of the form (2), take

$$\mathbf{B}(t) = \begin{cases} \langle 0,0,1\rangle & t \in [0,\pi/2] \\ \langle 1,0,0\rangle & t \in (\pi/2,\pi] \end{cases} \tag{5}$$

and $\gamma_0 = \langle 1,0,0\rangle$. Since for the corresponding $\tilde{\mathbf{B}}$, $e^{t\tilde{\mathbf{B}}}$ simply rotates all vectors about the vector $\mathbf{B}$ by t radians according to the right-hand rule, we see that $\gamma(\pi/2) = \langle 0,1,0\rangle$ and $\gamma(\pi) = \langle 0,0,1\rangle$. On the other hand,

$$\int_0^\pi dt\ \tilde{\mathbf{B}}(t) = \begin{pmatrix} \pi/2+1 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & \pi/2+1 \end{pmatrix}.$$

Immediately we see that $\int_0^\pi dt\ \tilde{\mathbf{B}}(t)\gamma_0 = \langle \pi/2+1, 1, 0\rangle \neq \langle 0,0,1\rangle$. What has happened here gives an opportunity to present a beautiful formula from the theory of groups. It is called the Campbell-Baker-Hausdorff formula and I leave it as an exercise to prove:

$$e^A e^B = e^C, \text{ where } C = A + B + \frac{1}{2}[A,B] + \ldots$$

where $[A,B] = AB - BA$. The fact that $e^A e^B \neq e^{A+B}$ is the reason our proposed method fails.

We now solve the general problem from first principles. Suppose we fix t and take N large enough so that on subintervals of $[0,t]$ of length t/N the matrix-valued function $\mathbf{A}$ is essentially constant. Then, over the subintervals $\{[kt/N,(k+1)t/N]\}_{k=0}^{N-1}$, the vector γ varies approximately according to the constant-$\mathbf{A}$ equations, (4), and we can chain the approximate solutions together to get

$$\begin{aligned}\gamma(t) &= \lim_{N\to\infty}(e^{\frac{t}{N}\mathbf{A}(t)})(e^{\frac{t}{N}\mathbf{A}((N-1)t/N)})(e^{\frac{t}{N}\mathbf{A}((N-2)t/N)})\ldots(e^{\frac{t}{N}\mathbf{A}(t/N)})(e^{\frac{t}{N}\mathbf{A}(0)})\gamma(0) \\ &= \lim_{N\to\infty}\left[\prod_{k=0}^{N-1}(e^{\frac{t}{N}\mathbf{A}(kt/N)})\right]\gamma(0).\end{aligned}$$

Now let us assume N arbitrarily large. In this case, each factor in the product $(e^{\frac{t}{N}\mathbf{A}(kt/N)}) \approx \mathbf{1} + \frac{t}{N}\mathbf{A}(kt/N)$ where $\mathbf{1}$ is the identity matrix (recall the Taylor expansion of the exponential function; the next term has $1/N^2$ in front). Thus[1]

$$\gamma(t) = \lim_{N\to\infty}\left[\prod_{k=0}^{N-1}(\mathbf{1} + \frac{t}{N}\mathbf{A}(kt/N))\right]\gamma(0),$$

which we may formally multiply out (grouping powers of t/N) as follows:

$$\begin{aligned}\gamma(t) = \lim_{N\to\infty}\Bigg[\mathbf{1} &+ \left(\frac{t}{N}\right)\sum_{k=0}^{N-1}\mathbf{A}(kt/N) + \left(\frac{t}{N}\right)^2\sum_{k>l}\mathbf{A}(kt/N)\mathbf{A}(lt/N) \\ &+ \left(\frac{t}{N}\right)^3\sum_{k>l>m}\mathbf{A}(kt/N)\mathbf{A}(lt/N)\mathbf{A}(lt/N) + \ldots\Bigg]\gamma(0).\end{aligned}$$

Taking the limit, we obtain time-ordered products

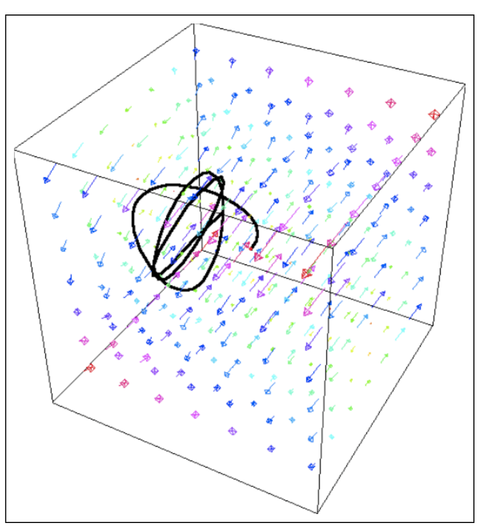
[1] Note that this is essentially mimicking how we made the calculation for continuously compounded interest in Differential Equations:

$$P(t) = P_0\lim_{N\to\infty}\left(1 + \frac{r}{N}\right)^{Nt} = P_0e^{rt}.$$

(6)
$$\gamma(t) = \left[\mathbf{1} + \int_0^t dt_1 \; \mathbf{A}(t_1) + \iint_{K_t^2} dt_1 dt_2 \; \mathbf{A}(t_1)\mathbf{A}(t_2) + \iiint_{K_t^3} dt_1 dt_2 dt_3 \; \mathbf{A}(t_1)\mathbf{A}(t_2)\mathbf{A}(t_3) + \dots \right] \gamma(0),$$
$$= \left[\mathbf{1} + \int_0^t dt_1 \; \mathbf{A}(t_1) + \int_0^t dt_1 \; \mathbf{A}(t_1) \int_0^{t_1} dt_2 \; \mathbf{A}(t_2) + \int_0^t dt_1 \; \mathbf{A}(t_1) \int_0^{t_1} dt_2 \; \mathbf{A}(t_2) \int_0^{t_2} dt_3 \; \mathbf{A}(t_3) + \dots \right] \gamma(0),$$
where $K_t^n = \{(t_1, t_2, \dots, t_n) \mid t > t_1 > t_2 > \dots > t_n > 0\} \subset \mathbb{R}^n$.

Exercises: When $\mathbf{A}$ *is constant, show that this series reduces to the solution* (4). *When does* $e^{\int^t \mathbf{A}(t)dt}$ *do what it was supposed to?*

Let us try our method out on an interpolated version of the vector $\mathbf{B}$ from (5): Let
$$\mathbf{B}(t) = \frac{\pi - t}{\pi} \langle 0, 0, 1 \rangle + \frac{t}{\pi} \langle 1, 0, 0 \rangle = \frac{1}{\pi} \langle t, 0, \pi - t \rangle \qquad t \in [0, \pi].$$
Then the corresponding $\tilde{\mathbf{B}}$ is
$$\tilde{\mathbf{B}}(t) = \frac{1}{\pi} \begin{pmatrix} 0 & t - \pi & 0 \\ \pi - t & 0 & -t \\ 0 & t & 0 \end{pmatrix},$$
leading to
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & t - \pi & 0 \\ \pi - t & 0 & -t \\ 0 & t & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (t - \pi)y \\ (\pi - t)x - tz \\ ty \end{pmatrix}.$$
Exercise: Prove that the length of γ, $x(t)^2 + y(t)^2 + z(t)^2$, *is a constant.* We compute some integrals now:
$$\int_0^t dt_1 \; \tilde{\mathbf{B}}(t_1) = \int_0^t dt_1 \begin{pmatrix} 0 & t_1 - \pi & 0 \\ \pi - t_1 & 0 & -t_1 \\ 0 & t_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & t^2/2 - \pi t & 0 \\ \pi t - t^2/2 & 0 & -t^2/2 \\ 0 & t^2/2 & 0 \end{pmatrix}$$
$$\int_0^t dt_1 \int_0^{t_1} dt_2 \; \tilde{\mathbf{B}}(t_1)\tilde{\mathbf{B}}(t_2) = \begin{pmatrix} \frac{-1}{8} t^2 (t - 2\pi)^2 & 0 & \frac{1}{24} t^3 (4\pi - 3t) \\ 0 & \frac{-1}{4} t^2 (2\pi^2 - 2\pi t + t^2) & 0 \\ \frac{1}{24} t^3 (4\pi - 3t) & 0 & \frac{-1}{8} t^4 \end{pmatrix}$$
$$\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \; \tilde{\mathbf{B}}(t_1)\tilde{\mathbf{B}}(t_2)\tilde{\mathbf{B}}(t_3) =$$
$$\begin{pmatrix} 0 & \frac{1}{120} t^3 (20\pi^3 - 30\pi^2 t + 18\pi t^2 - 5t^3) & 0 \\ \frac{1}{120} t^3 (-20\pi^3 + 30\pi^2 t - 23\pi t^2 + 5t^3) & 0 & \frac{1}{120} t^4 (5\pi^2 - 7\pi t + 5t^2) \\ 0 & \frac{-1}{120} t^4 (15\pi^2 - 12\pi t + 5t^2) & 0 \end{pmatrix}$$
and so on. I have included the *Mathematica* code computing these matrices below. Here is a frame from the integration:



Exercise: Mimic and generalize the procedure by which a first-order linear problem is used to solve a second order problem via the equivalence

$$y'' + p(t)y' + q(t)y = 0; \quad y(0) = y_0,\ y'(0) = y_1 \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -p & -q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}; \quad y(0) = y_0,\ x(0) = y_1.$$

Answer: Let P, Q be arbitrary matrix-valued functions of t. To solve

$$\gamma'' + P(t)\gamma' + Q(t)\gamma = \mathbf{0}; \quad \gamma(0) = \gamma_0,\ \gamma'(0) = \gamma_1$$

set up the system

$$\begin{pmatrix} \xi' \\ \gamma' \end{pmatrix} = \begin{pmatrix} -P & -Q \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \xi \\ \gamma \end{pmatrix}; \quad \gamma(0) = \gamma_0,\ \xi(0) = \gamma_1.$$

Also, figure out how to deal with inhomogeneous problems. One suggestion would be to use the method of variation of parameters for the second-order problem and translate it to the first-order system.
Problem: In [BD], *study Chapter 7. You will see that the eigenvalues of* $\mathbf{A}$, *when it is constant, govern the behavior of the solutions of the system. What happens when* $\mathbf{A}$ *varies in time?*
Problem: In Chapters II and III of [ZT], *there are several methods presented for solving integral curve problems. Relate these to the current method.*
Problem: In [R], *Chapter II and* [FH], *Chapters 1-3, the Feynman path integral formulation of Quantum Mechanics is described. This picture uses a similar idea to the one above: locally potentials are constant and so the wave function propagates as if it were that of a free particle. Make connections to the current method and find the appropriate generalizations of the Gaussian integrals from Feynman's Quantum Mechanics.*

Here is the (inflexible and uncommented) *Mathematica* code that computed the integrals for a similar but periodically varying magnetic field $\mathbf{B}(t) = \langle \sin(t), 0, \cos(t) \rangle$, and produced the animation.

```
<<Graphics‘PlotField3D‘

B[t_] := {{0,-Cos[t],0},{Cos[t],0,-Sin[t]},{0,Sin[t],0}}
Init := {1,0,0}
timescale := 8
tmax := 80
kmax := 20

Do[
   gdd[t] = PlotVectorField3D[
        (1/7)B[t/timescale].{x,y,z},
        {x,-2,2},
        {y,-2,2},
        {z,-2,2},
        VectorHeads->True,
        ScaleFactor->False,
        Lighting->False,
        ColorFunction->Hue,
        ImageSize->500,
        BoxRatios->{1,1,1}
   ],
   {t,0,tmax}
];

A[0, t_] := {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
Do[
```

```
   A[k + 1, t_] = Integrate[B[t0].A[k, t0], {t0, 0, t}],
   {k, 0, 10}
]

V[1] = {0, 0, 0};
Do[
   V[k + 1] = A[k, t].{-1, 0, 0} + V[k],
   {k, 1, 10}
]


f[s_] = V[kmax+1][[1]]+Init[[1]]/.t->s;
g[s_] = V[kmax+1][[2]]+Init[[2]]/. t->s;
h[s_] = V[kmax+1][[3]] +Init[[3]]/. t->s;

Do[
   gddd[t] = ParametricPlot3D[
      {f[s/timescale],g[s/timescale],h[s/timescale],{Thickness[.008]}},
      {s,.01,t},
      BoxRatios ->{1,1,1},
      ViewPoint->{1, .7, 1.2}
   ],
   {t,0,tmax}
];

Do[
   {Print[N[f[t/timescale]^2+g[t/timescale]^2+h[t/timescale]^2] ,  t/timescale]},
   {t,0,tmax}
]

Do[
   {Show[
      gdd[t],
      gddd[t],
      BoxRatios->{1,1,1},
      ImageSize->500
   ],
   Print[t]},
   {t,0,tmax}
]
```

References

[BD] Boyce, W.E., DiPrima, R.C.: *Elementary Differential Equations and Boundary Value Problems*, Wiley

[FH] Feynman, R.P., Hibbs, A.R.: *Quantum Mechanics and Path Integrals*, McGraw-Hill

[GSU] http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/larmor.html

[R] Ramond, P.: *Field Theory, A Modern Primer*, Frontiers in Physics Series, Benjamin/Cummings

[ZT] Zachmanoglou, E.C., Thoe, D.W.: *Introduction to Partial Differential Equations with Applications*, Dover